Lecture 7

Examples: Characteristics

Consider DE

$$x_2 u_{x_1x_1} + u_{x_2x_2} = 0$$

Characteristic form

$$C(x,\xi) = x_2\xi_1^2 + \xi_2^2$$

so the characteristic cone:

$$\{\xi \ : \ C(x,\xi) = 0\}$$

That is

$$\left(\frac{\xi_2}{\xi_1}\right)^2 = -x_2$$

The vector ξ is in fact $\bigtriangledown\phi$ hence condition

$$C(x, \underbrace{\bigtriangledown\phi}_{normal\ to\ S}) = 0$$

Last time

We were looking at semi-linear equation

$$\sum \alpha_i(x)\partial_i u = \beta(x,u) \tag{1}$$

Theorem 1. $\alpha,\beta,\gamma,g \in C^k$. *Suppose* Γ *nowhere characteristic at each point* $\xi \in \Gamma$

$$\gamma_\xi(t) = \alpha(\gamma_\xi(t)), \qquad \gamma_\xi(0) = \xi$$

$$v_\xi(t) = \beta(\gamma_\xi(t)), v_\xi(t)), \qquad v_\xi(0) = g(\xi)$$

$\exists\Sigma \subset \Gamma \times \mathbb{R}$ *open*, $\Sigma \supset \Gamma \times \{0\}$ *s.t*

$$u(\gamma_\xi(t)) = v_\xi(t), \qquad (\xi,t) \in \Sigma,$$

solves (1) *in* $U = \{\gamma_\xi(t) \ : \ (\xi,t) \in \Sigma\}$, *and* $u\Big|_\gamma = g$. *Any differentiable solution of* (1) *wit I.C* g *coincides with* $u \in U$

Proof sketch. Let $\varphi(\xi,t) = \gamma_\xi(t)$

$$\gamma_\xi(0) = \xi, \qquad \gamma_{\gamma_\xi(t)}(-t) = \xi$$

More generally

$$\gamma_{\gamma_\xi(t)}(s) = \gamma_\xi(t+s)$$

We aim to prove the existence of an inverse map locally $\varphi : x \mapsto t(x)$ by means of the Inverse Function Theorem. This is asserted by showing determinant $|D\varphi(\xi,0)| \neq 0$ where $\varphi(\xi,t) : \ \Gamma \times \mathbb{R} \mapsto \mathbb{R}^n$ defined by $\varphi(\xi,t) := \gamma_\xi(t)$. It is important to know that in the choice $\xi \in \Gamma \subseteq \mathbb{R}^{n-1}$, meanwhile the

Class notes by Ibrahim Al Balushi

ξ in argument of φ is not restricted to Γ. We now calculate derivatives of γ_ξ to compute the total directional derivative of φ so we have

$$D\varphi = \left[\left\{ \frac{\partial \gamma_\xi}{\partial \xi_i} \right\}_{i=1}^{n-1}, \frac{\partial \gamma_\xi}{\partial t} \right]_{n\times n} \tag{2}$$

We use the linear approximation for the directional derivative of γ_ξ with respect to the ξ direction (See h.w 1 question 7).
$\exists A \in \mathbb{R}^2,\ a \in \mathbb{R}\ s.t$

$$\gamma_{\xi+\dot{\xi}}(t) = \xi + A\dot{\xi} + at + o(|\dot{\xi}| + |t|)$$

since γ is differentiable $A = D_\xi \gamma$

$$\gamma_{\xi+\dot{\xi}}(0) = \xi + D_\xi\gamma(\xi,0)\dot{\xi} + o(|\dot{\xi}|) \implies D_\xi\gamma = I_{n\times n}$$

$$and\ \frac{\partial \gamma_\xi}{\partial t}(0) = \alpha(\xi)$$

Notice from (2) that we only need the first $n-1$ components of $D_\xi\gamma$ and the last column is determined by the expression above $\alpha(\xi)$

$$D\varphi = \underbrace{\left[I_{n-1\times n-1} \middle| \alpha(\xi) \right] \dot{\xi}}_{invertible}$$

$\square$

Quasilinear 1st Order Eq

$$\sum^n \alpha_i(x,u)\partial_i u = \alpha_{n+1}(x,u)$$

$$\tilde{\Omega} \subset \mathbb{R}^{n+1}, \qquad \alpha :\ \tilde{\Omega} \mapsto \mathbb{R}^{n+1}$$

Graph characteristics $\gamma :\ I \mapsto \mathbb{R}^{n+1} \qquad I \subset \mathbb{R},\ open\ interval$

$$\gamma'(t) = \alpha(\gamma(t))$$

$$u(\gamma_{\xi,1}(t), ..., \gamma_{\xi,n}(t)) = \gamma_{\xi,n+1}(t). \qquad \gamma_\xi(0) = (\xi, g(\xi))\ \xi \in \Gamma.$$

same theorem above holds.

Classic Example — Traffic Equation

$$u_t + uu_x = 0$$

Scalar conservation laws:

$$u(x,t)$$

$$f\ :\ \mathbb{R} \mapsto \mathbb{R}^n$$

$$\frac{\partial}{\partial t}\int_\Omega u = -\int_{\partial\Omega} f(u)v = -\int_\Omega div\ f(u)$$

$f(u) \in C^1 \implies u_t + div\ f(u)$. We have

$$u_t + f(u)_x = u_t + f'(u)u_x = 0$$

From the diagram we have

$$t\alpha_1 - t\alpha_2 = l \implies \frac{1}{t} = -\frac{\alpha_2 - \alpha_1}{l} = -f'(u)_x$$

since $\alpha_i = f'(u(x_i))$.

$$1/t = -min\ f'(u)_x = min\ f''(u)u_x = -min f''(g)g'$$

is when we have wave breaking.